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ON RIGIDITY OF GRAUERT TUBES OVER HOMOGENEOUS RIEMANNIAN MANIFOLDS

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Abstract.

Given a real-analytic Riemannian manifold X there is a canonical complex structure, which is compatible with the canonical complex structure on T^*X and makes the leaves of the Riemannian foliation on TX into holomorphic curves, on its tangent bundle. A *Grauert tube* over X of radius r, denoted as T^rX, is the collection of tangent vectors of X of length less than r equipped with this canonical complex structure.

In this article, we prove the following two rigidity property of Grauert tubes. First, for any real-analytic Riemannian manifold such that $r_{max} > 0$, we show that the identity component of the automorphism group of T^rX is isomorphic to the identity component of the isometry group of X provided that $r < r_{max}$. Secondly, let X be a homogeneous Riemannian manifold and let the radius $r < r_{max}$, then the automorphism group of T^rX is isomorphic to the isometry group of X and there is a unique Grauert tube representation for such a complex manifold T^rX.

1. Introduction.

The purpose of this article is to give an affirmative answer to the rigidity of Grauert tubes over (compact or non-compact) homogeneous Riemannian spaces. On the way to prove this, we are also able to show that for any real-analytic Riemannian manifold, the identity component of the automorphism group of a Grauert tube is isomorphic to the identity component of the isometry group of the center when it is defined and the radius is not the critical one.

It was observed by Grauert [6] that a real-analytic manifold X could be embedded in a complex manifold as a maximal totally real submanifold. One way to see this is to complexify the transition functions defining X. However, this complexification is not unique. In [7] and [12], Guillemin-Stenzel and independently Lempert-Szőke encompass certain conditions on the ambient complex structure to make the complexification canonical for a given real-analytic Riemannian manifold.

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In short, they were looking for a complex structure, on part of the cotangent bundle T^*X, compatible with the canonical symplectic structure on T^*X. Equivalently, it is to say that there is a unique complex structure, called the adapted complex structure, on part of the tangent bundle of X making the leaves of the Riemannian foliation on TX into holomorphic curves. The set of tangent vectors of length less than r equipped with the adapted complex structure is called a Grauert tube T^rX.

Since the adapted complex structure is constructed canonically associated to the metric g of the Riemannian manifold X, the differentials of the isometries of X are automorphisms of T^rX. Conversely, it is interesting to see whether all automorphisms of T^rX come from the differentials of the isometries of X or not. When the answer is affirmative, we say the Grauert tube is *rigid*.

With respect to the adapted complex structure, the length square function $\rho(x,v) = |v|^2, v \in T_xX$, is strictly plurisubharmonic. When the center X is compact, the Grauert tube T^rX is exhausted by ρ, hence is a Stein manifold with smooth strictly pseudoconvex boundary when the radius is less than the critical one. In this case, the automorphism group of T^rX is a compact Lie group. Base on these, Burns, Burns-Hind ([1], [2]) are able to prove the rigidity for Grauert tubes over compact real-analytic Riemannian manifolds. In [1], Burns also showed that the rigidity of a Grauert tube is equivalent to the uniqueness of center when the Grauert tube is constructed over a compact manifold.

When X is non-compact nothing particular is known, not even to the general existence of a Grauert tube over X. When it exists, most of the good properties in the compact cases are lacking here since the length square function ρ is no longer an exhaustion. However, in this article we are able to prove some kind of weak rigidity for general Grauert tubes. That is, we are able to show that $Aut_0(T^rX) = Isom_0(X)$ when it is defined and $r < r_{max}$.

By now, there are very few complete non-compact cases we are sure about the existence of Grauert tubes. One kind of them are those over co-compact real-analytic Riemannian manifolds, the Grauert tubes are simply the lifting of the Grauert tubes over their compact quotients. The second kind are Grauert tubes over real-analytic homogeneous Riemannian manifolds. In [10], Kan and Ma have proved the rigidity of Grauert tubes over (compact or non-compact) symmetric spaces based on the co-compactness of symmetric spaces. Later on, in [9], the author proves that the uniqueness of such Grauert tubes also holds.

We prove in this article that not only the rigidity but also the uniqueness holds for

Grauert tubes over real-analytic homogeneous Riemannian manifolds. The center manifold could be either compact or non-compact. Moreover, it is also true for Grauert tubes constructed over quotient manifolds of a homogeneous Riemannian space. The main result is Theorem 7.6. The condition $r < r_{max}$ is necessary since Grauert tubes over non-compact symmetric spaces of rank one of maximal radius were proved in [3] to be non-rigid.

On the way to this we, omitting the requirement on the tautness, prove a generalized version of the Wong-Rosay theorem on the characterization of ball to any domain in a complex manifold on Theorem 3.1. We also prove at Theorem 5.2 on the complete hyperbolicity of Grauert tubes over homogeneous Riemannian manifolds. On Theorem 6.4, a weak version of the rigidity is given for any general Grauert tubes.

The outline of this article is as follows. In §2, the behavior of the Kobayashi metric near C^2 strictly pseudoconvex boundary points was observed. §3 is devoted to the generalized Wong-Rosay theorem. We give a brief review on Grauert tubes and prove one of the key theorem, Theorem 4.1, toward the rigidity in §4. In §5, the complete hyperbolicity of Grauert tubes over homogeneous Riemannian spaces was claimed. §6 is on the discussion of the equality of $Aut_0(T^rX)$ and $Isom_0(X)$. In §7, we prove the rigidity $Isom(X) = Aut(T^rX)$ and the uniqueness of Grauert tubes.

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2. Estimates of Kobayashi metrics near a strictly pseudoconvex point.

For complex manifolds N and M, let $Hol(N, M)$ be the family of holomorphic mappings from N to M. Let Δ denote the unit disk in $\mathbb{C}$. The *Kobayashi pseudometric* on M is defined by

$$F_M(z, \xi) = \inf\{\alpha|\ \alpha > 0, \exists f \in H(\Delta, M), f(0) = z, f'(0) = \frac{\xi}{\alpha}\}.$$

The Kobayashi pseudometric F_M is holomorphic decreasing, i.e., if $f : N \to M$ is a holomorphic map, then $F_M(f(z), f_*\xi) \leq F_N(z, \xi)$. We also quote the following

property (c.f. p.128, Proposition 2, [15]) for later use: if K is a compact subset of M contained in a coordinate polydisk, then there is a constant C_K such that

$$F_M(z,\xi) \le C_K\|\xi\| \tag{2.1}$$

for all $z \in K$ where $\|\xi\| = max|\xi_i|$.

Royden [15] has shown that the *Kobayashi pseudodistance* d^K_M is the integrated form of F_M. That is, given $z, w \in M$,

$$d^K_M(z,w) = \inf_\gamma \int_0^1 F_M(\gamma(t), \gamma'(t))dt$$

where the infimum is taken over all piecewise C^1-smooth curves $\gamma : [0,1] \to M$ joining the two points $z, w \in M$.

The manifold M is said to be *hyperbolic* if its Kobayashi pseudodistance is a distance, i.e., $d^K_M(z,w) = 0$ implies $z = w$. The hyperbolicity of a manifold M is equivalent to the following (c.f.[15]): for any $x \in M$ there exists a neighborhood U of x and a positive constant c such that $F_M(y,\xi) \ge c\|\xi\|$ for all $y \in U$. In this case the Kobayashi pseudodistance is called the Kobayashi distance of M. It is well known that a bounded domain in $\mathbb{C}^n$ is hyperbolic. Furthermore, Sibony [16] proved that a complex manifold with a bounded strictly plurisubharmonic function must be hyperbolic. A hyperbolic manifold M is said to be *complete hyperbolic* if its Kobayashi metric is complete, i.e., if each Cauchy sequence in the Kobayashi metric has a limit or equivalently that any ball of finite radius is relatively compact in M.

In [5], Graham has observed the boundary behavior of the Kobayashi metric on bounded strictly pseudoconvex domains $D \subset \mathbb{C}^n$. He showed that the boundary is at infinite distance, i.e., $\lim_{q\to\partial D} d^K_D(q,K) = \infty$ for any compact $K \subset D$. He hence concluded that every bounded strictly pseudoconvex domain in $\mathbb{C}^n$ is complete hyperbolic.

For the rest of this section, we would like to make a similar estimate for C^2-smooth strictly pseudoconvex boundary points of a domain in a complex manifold.

Lemma 2.1. *Let D be a domain in a complex manifold M and let $q \in \partial D$ be a C^2-smooth strictly pseudoconvex boundary point. Then there exist a domain $D' \supset D$ and a bounded plurisubharmonic function h on D' which is strictly plurisubharmonic on a neighborhood of q.*

Proof. Without loss of generality, we may assume that there is a neighborhood U of q in M and a coordinate chart Φ from U onto an open set $V \subset \mathbb{C}^n$ such that $\Phi(q) = 0$ and the local defining function ρ for D at q has the following form:

$$\rho \cdot \Phi^{-1}(z) = 2Re\ z_n + |z|^2 + o(|z|^2).$$

Let

$$\tilde{f}(z) = 2Re\ z_n + \frac{1}{2}|z|^2 + o(|z|^2)$$

be a plurisubharmonic function on V. Suitably shrinking the neighborhood U, we see that the function f defined as $f = \tilde{f} \cdot \Phi$ is a bounded plurisubharmonic function on U which is strictly plurisubharmonic near the point q.

Denoting $U^- = U \cap D$, $V^- = \Phi(U^-)$ and $W^- = \{w \in U : f(w) < 0\}$, then $U^- \subsetneq W^-$ and $f^{-1}(0) \cap \rho^{-1}(0) = \{q\}$. Therefore,

$$m := \max_{w \in \overline{D} \cap (\partial U)} f(w) < 0.$$

Let $D' = D \cup U$. We define a function h on D' as following:

$$h = \begin{cases} \max(f, \frac{m}{2}) & \text{on} \quad U, \\ \dfrac{m}{2} & \text{on} \quad D - U. \end{cases}$$

It is clear that the domain D' and the function h fulfill the requirement in the statement of the lemma. $\square$

Applying Lemma 2.1 to Theorem 3 of [16], there exist a neighborhood U of q in D' and a positive constant $c > 0$ such that

$$F_{D'}(z, \xi) \geq c\|\xi\| \quad \text{for all } z \in U, \xi \in \mathbb{C}^n. \tag{2.2}$$

By the holomorphic decreasing of the Kobayashi pseudodistance, we have

$$F_D(z, \xi) \geq c\|\xi\| \quad \text{for all } z \in U \cap D, \xi \in \mathbb{C}^n, \tag{2.3}$$

which concludes that M is hyperbolic on $U \cap D$.

It was first mentioned by Royden in [15] without proof and was later on proved by Graham at Lemma 4, p.234 of [5] that if $P \subset G$ are two complex manifolds (the original requirement on the hyperbolicity of the manifolds could be removed away) then, for $z \in P$,

$$F_P(z, \xi) \leq \coth(\inf_{w \in G - P} D^*_G(z, w)) F_G(z, \xi). \tag{2.4}$$

$D^*_G(z,w)$ is defined as follows:

$$D^*_G(z,w) = \inf\{\kappa(a,b)|\ \exists f \in H(\Delta, G), f(a) = z, f(b) = w\}$$

where κ is the Poincaré distance for Δ. From the equivalent definition of the Kobayashi pseudodistance in the classical sense, it is clear that $d^K_G(z,w) \le D^*_G(z,w)$. Making a little modification on this, we are able to prove the following:

Lemma 2.2. *Let D be a domain in a complex manifold M and let $q \in \partial D$ be a C^2-smooth strictly pseudoconvex boundary point. If V is a neighborhood of q in M, then*

$$\lim_{p\in D, p\to q} d^K_D(p, D\backslash V) = \infty.$$

Proof. Let the domain D' be constructed as in Lemma 2.1. By (2.2) there exist a neighborhood U of q in D' and a positive constant c such that $F_{D'}(z,\xi) \ge c\|\xi\|$ for all $z \in U$. Let $V \Subset U$ be a smaller neighborhood of q in D'. Then there exists a $\delta > 0$ such that $d^K_{D'}(z,w) \ge \delta$ for all $z \in V$ and $w \in D' - U$. By the holomorphic decreasing property of the Kobayashi pseudodistance

$$d^K_D(z,w) \ge \delta \quad \text{for all } \ z \in V \cap D,\ w \in D - U.$$

For all $z \in V \cap D$,

$$\inf_{w\in D-U} D^*_D(z,w) \ge \inf_{w\in D-U} d^K_D(z,w) \ge \delta > 0,$$

and

$$\coth(\inf_{w\in D-U} D^*_D(z,w)) \le coth(\delta) \le L$$

for some constant L independent of z.

On the other hand, $U \cap D$ could also be viewed as a bounded domain $\tilde{U}$ in $\mathbb{C}^n$ with C^2-smooth strictly pseudoconvex boundary points through the biholomorphic map Φ. We set $E = \Phi(V \cap D)$. By (2.4),

$$F_D(z,\xi) \ge \frac{1}{L} F_{U\cap D}(z,\xi) = \frac{1}{L} F_{\tilde{U}}(\Phi(z), \Phi_*(\xi)) \tag{2.5}$$

for all $z \in V \cap D$. The completeness of $d^K_{\tilde{U}}$ along with (2.5) leads to,

$$\begin{aligned} \lim_{p\in D, p\to q} d^K_D(p, D\backslash V) &\ge \lim_{p\in V, p\to q} d^K_D(p, U\backslash V) \\ &\ge \frac{1}{L} \lim_{p\in V, p\to q} d^K_{\tilde{U}}(\Phi(p), \tilde{U}\backslash E) \\ &= \infty. \end{aligned}$$

□

3. An extension of the Wong-Rosay Theorem.

Historically speaking, Wong proved the biholomorphic equivalence of a bounded strictly pseudoconvex domain with non-compact automorphism group and the unit ball in $\mathbb{C}^n$ in 1977. Klembeck presented in 1978 a completely different proof which is much more differential geometric. Then in 1979, Rosay strengthened the theorem to a bounded domain such that there exists a sequence of automorphisms $\{f_j\}$ sending a point p to a smooth strictly pseudoconvex boundary point. Efimov [4], using Pichuk's rescaling method along with certain estimate on the Kobayashi metric, has extended the above theorem of Rosay to unbounded domains in $\mathbb{C}^n$. In an attempt to solve the rigidity for Grauert tubes, the author and Ma [10] also proved an extended version of the Wong-Rosay theorem to certain complete hyperbolic Stein manifolds. In this section we will drop out the requirement on the complete hyperbolicity and prove a version of Wong-Rosay theorem for general domains inside complex manifolds.

Theorem 3.1. *Let M be a domain in a complex manifold $\hat{M}$. Suppose that there exist a point $p \in M$ and a sequence of automorphisms $\{f_j\} \subset Aut(M)$ such that $f_j(p) \to q \in \partial M$, a C^2-smooth strictly pseudoconvex point. Then M is biholomorphic to the unit ball.*

Proof.

Let U be a connected neighborhood of q in $\hat{M}$ such that there exists a biholomorphic map Φ from U to a neighborhood $\Phi(U)$ of 0 in $\mathbb{C}^n$ which maps q to the origin and such that the Kobayashi pseudodistance d_M^K is a distance in U. Since q is a C^2-smooth strictly pseudoconvex point, we could choose suitable Φ and U so that $\Phi(U) = E$ is a strictly convex domain in $\mathbb{C}^n$ and the domain

$$D = \Phi(U \cap M) = \{z \in E : \phi(z) < 0)\} \subset \mathbb{C}^n$$

is strictly pseudoconvex with the defining function

$$\phi(z) = \operatorname{Re} z_n + |'z|^2 + \operatorname{Re} \sum_{j=1}^{n} b_j z_j \bar{z}_n + o(|z|^2)$$

where $z = ('z, z_n)$ and $b_j \in \mathbb{C}$. Let

$$p_j = f_j(p), \quad 2r_j = d_M^K(p_j, M \backslash U).$$

From Lemma 2.2, we see that $r_j \to \infty$ as $j \to \infty$. Let

$$B_K(p, r) = \{w \in M : d_M^K(p, w) < r\}$$

denote the Kobayashi ball of radius r centered at p. Then

$$f_j(B_K(p,r_j)) = B_K(p_j,r_j) \subset U \cap M.$$

Hence,

$$K_j = \overline{B_K(p,r_j)} \subset f_j^{-1}(U \cap M). \tag{3.1}$$

For any $z \in M$, there exists a neighborhood $B_K(p,r_j)$ of z and a $f_j \in Aut(M)$ sending $B_K(p,r_j)$ to $U \cap M$. It follows from the biholomorphic invariance of F_M and (2.3) that

$$F_M(z,\xi) = F_M(f_j(z), f_{j*}\xi) \geq c\|\xi\| \quad \text{for all } z \in B_K(p,r_j),\ \xi \in \mathbb{C}^n.$$

Hence, every point of M is a hyperbolic point, i.e., M is in fact a hyperbolic manifold.

Let ζ_j denote the unique point on $L = \Phi(\partial M \cap U)$ closest to $a_j = \Phi(p_j)$ in the Euclidean distance. Let g_j denote the composition of the translation that maps ζ_j to 0 and a unitary transformation, taking the tangent plane $T_{\zeta_j}L$ to the plane $\{\text{Re } z_n = 0\}$. The mapping g_j sends D biholomorphically onto a bounded domain G_j with the defining function given by

$$\begin{aligned}\phi_j &= \phi \circ g_j^{-1}(z)\\ &= c_j \text{Re } z_n + d_j|'z|^2 + \text{Re} \sum_{k=1}^{n-1} b_k \mu_k^j \bar{c}_j z_k \bar{z}_n + \text{Re } b_n |c_j|^2 |z_n|^2 + 0(|z|^2)\end{aligned}$$

where

$$c_j \to 1, \quad d_j \to 1, \quad \mu_k^j \to 1, \text{ as } j \to \infty.$$

Since a_j lies on the normal to L at ζ_j, the point

$$g_j(a_j) = ('0, -\delta_j).$$

By construction, $a_j \to 0$ and hence, $\zeta_j \to 0$ and $\delta_j \to 0$ as $j \to 0$. Let F_j be the scaling function defined by

$$F_j('z, z_n) = ('z/\sqrt{\delta_j}, z_n/\delta_j),$$

and let $h_j = F_j \circ g_j$. Then $h_j(a_j) = ('0,-1)$ and h_j maps D biholomorphically to a domain $H_j \subset \mathbb{C}^n$ with the defining function

$$\begin{aligned}\psi_j(w) &= (1/\delta_j)\phi \circ h_j^{-1}(w)\\ &= (1/\delta_j)\phi \circ g_j^{-1} \circ F_j^{-1}(w)\\ &= (1/\delta_j)\phi_j(\sqrt{\delta_j}\ 'w, \delta_j w_n)\\ &= c_j \text{Re } w_n + d_j|'w|^2 + \text{Re } \delta_j^{\frac{1}{2}} \sum_{k=1}^{n-1} b_k \mu_k^j \bar{c}_j w_k \bar{w}_n\\ &\quad + \text{Re } \delta_j\ b_n |c_j|^2 |w_n|^2 + \delta_j^{\varepsilon} 0(|w|^2), \text{ for some } \varepsilon > 0.\end{aligned}$$

The defining functions ψ_j converge, uniformly on compact subsets of H_j, to $\psi(w) := \text{Re } w_n + |'w|^2$ which is the defining function of the Siegel half-space $\mathcal{U}$, biholomorphically equivalent to the ball. Let

$$R_\nu = \{w \in \mathbb{C}^n : \text{Re } w_n + (1-\nu)|'w|^2 < 0\}.$$

These domains satisfy $R_{\nu_1} \subset R_{\nu_2}$ for $\nu_1 < \nu_2$. Let

$$\nu_j = \inf\{\nu > 0 : H_j \subset R_\nu\}.$$

Then $\nu_j \to 0$ as $j \to \infty$. Without loss of generality we assume that $\nu_j < 1/2$.

Consider the maps

$$\Psi_j = h_j \circ \Phi \circ f_j : \ \ f_j^{-1}(U \cap M) \to H_j \subset R_{\nu_j}.$$

By (3.1), each compact subset of M is contained in $f_j^{-1}(U \cap M)$ for sufficiently large j. Since the domains R_{ν_j} are contained in $R_{1/2}$, which is biholomorphic to the unit ball in $\mathbb{C}^n$. By Montel's theorem, there exists some subsequence $\{\Psi_{j_\nu}\}$ converges in the compact open topology to a holomorphic map $\Psi : M \to R_{1/2}$. As $\nu_j \to 0$ we conclude that $\Psi(M) \subset R_\nu$ for each $0 < \nu < \frac{1}{2}$, hence $\Psi : M \to \mathcal{U}$ is a holomorphic map.

We would like to show that Ψ is actually a biholomorphism. Following the idea of Efimov [4], we first show that Ψ is a locally one-one map.

Since $\Psi_j(p) = ('0, -1)$ for every j and $H_j \to \mathcal{U}$ as $j \to \infty$, we may assume that there exists a positive $\beta < \frac{1}{2}$ such that the Euclidean ball $\hat{B} := B(('0,-1), \beta)$ is contained in all $\Psi_j(K_j) \cap \mathcal{U}$ for j sufficiently large. Since M is hyperbolic, $\overset{\circ}{K}_j$, the interior of K_j defined at (3.1), is hyperbolic as well. By the hyperbolicity of M, the holomorphic decreasing and the invariance property of a hyperbolic metric, we have

$$\begin{aligned} c_1\|\xi\| \le F_M(p,\xi) &\le F_{\overset{\circ}{K}_j}(p,\xi) \\ &= F_{\Psi_j(\overset{\circ}{K}_j)}(('0,-1), d\Psi_j|_p(\xi)) \\ &\le F_{\hat{B}}(('0,-1), d\Psi_j|_p(\xi)) \\ &\le c_2\|d\Psi_j|_p(\xi)\|; \end{aligned}$$

the last inequality comes from (2.1). Hence there are constants c_1 and c_2 independent of j such that

$$\|d\Psi_j|_p(\xi)\| \ge \frac{c_1}{c_2}\|\xi\|. \tag{3.2}$$

Since Ψ_j are biholomorphisms, $d\Psi_j$ are nowhere zero for every j. The Hurwitz's theorem implies that the limit function $d\Psi$ is either nowhere zero or identically zero. The uniform estimate of (3.2) gives that $d\Psi|_p \neq 0$. Hence $d\Psi$ is nowhere zero and Ψ is locally one-one.

We claim that it is in fact globally one-one. Suppose there exist $x, y \in M$ such that $\Psi(x) = \Psi(y) = s \in \mathcal{U}$. Then there is a Euclidean ball $\mathcal{V} = B(s, l) \subset \mathcal{U}$ and two disjoint neighborhoods V_x and V_y of x and y in M respectively such that both $\overline{V}_x$ and $\overline{V}_y$ are biholomorphic to $\overline{\mathcal{V}}$ through Ψ. Choose j sufficiently large so that $|\Psi_j - \Psi| < \frac{\ell}{2}$ in $V_x \cup V_y$. The boundary of the connected domain $\Psi_j(V_x)$ is less than $\frac{l}{2}$ away from the boundary of $\mathcal{V}$ and $|\Psi_j(x) - s| < \frac{l}{2}$. Thus $s \in \Psi_j(V_x)$. Similarly, $s \in \Psi_j(V_y)$. That is $\Psi_j(V_x) \cap \Psi_j(V_y) \neq \emptyset$ which contradicts to the injectivity of Ψ_j.

Hence Ψ is a one-one holomorphic map, M is biholomorphic to its image $\Psi(M) \subset \mathcal{U}$. As $\mathcal{U}$ is biholomorphic to the unit ball in $\mathbb{C}^n$, we view $M \approxeq \Psi(M)$ as a bounded domain in $\mathbb{C}^n$ where the Montel's theorem applies.

Consider the map

$$\Psi_j^{-1} : H_j \to f_j^{-1}(U \cap M) \subset M.$$

There exists a subsequence $\Psi_{\nu_j}^{-1}$ converges to a holomorphic map

$$\hat{\Psi} : \mathcal{U} \to M.$$

From the construction, it is clear that $\hat{\Psi}$ is the inverse of Ψ. Hence Ψ is a biholomorphism from M to the ball $\mathcal{U}$. $\square$

4. General property of $Aut(T^rX)$.

Let (X, g) be a real-analytic Riemannian manifold. The *Grauert tube*

$$T^rX = \{(x, v) : x \in X, v \in T_xX, |v| < r\}$$

is the collection of tangent vectors of length less than r equipped with the unique complex structure, the *adapted complex structure*, which turns each leaves of the Riemannian foliation on $T^rX \backslash X$ into holomorphic curves. That is, the map $f(\mu + i\tau) = \tau\gamma'(\mu)$, $\mu + i\tau \in \mathbb{C}$, is holomorphic, whenever it is defined, for any geodesic γ of X. There is a natural anti-holomorphic involution σ fixing every point of the center,

$$\sigma : T^rX \to T^rX, \quad (x, v) \to (x, -v).$$

The length square function defined as $\rho(x, v) = |v|^2$ is strictly plurisubharmonic and satisfies the complex homogeneous Monge-Ampère equation $(dd^c\rho)^n = 0$ on $T^rX\backslash X$. The initial condition for ρ is that $\rho_{i\bar{j}}|_X = \frac{1}{2}g_{ij}$. We usually called the set (T^rX, X, g, ρ) a *Monge-Ampère model* ; X the center and r the radius.

For each real-analytic Riemannian manifold, there exists a $r_{max}(X) \geq 0$, maximal radius, such that the adapted complex structure is well-defined on $T^{r_{max}}X$ whereas it blows up on T^sX for any $s > r_{max}$. When the center manifold X is compact, it is clear that $r_{max}(X) > 0$ since we could always paste the locally defined adapted complex structure together. For non-compact X, it is very likely that the maximal possible radius degenerates to zero. However, when (X, g) is homogeneous the corresponding $r_{max}(X)$ is always positive. The same holds for co-compact X. The Kähler manifold T^rX is a submanifold of $T^{r_{max}}X$ for any $r < r_{max}(X)$.

It is clear from the construction of Grauert tubes that the differential of an isometry of (X, g) is actually an automorphism of T^rX, i.e., $Isom(X) \subset Aut(T^rX)$. We ask whether the converse holds or not.

If $Isom(X) = Aut(T^rX)$, i.e., every automorphism comes from the differential of an isometry of X, we say the Grauert tube T^rX has the *rigidity* property. In the compact case, this is equivalent to the uniqueness of center: there is no other Riemannian manifold (Y, h) such that the complex manifold T^rX is represented as T^rY, a Grauert tube over Y.

When the center X is compact, the rigidity of Grauert tubes has been completely solved by Burns-Hind in [2]. They prove that $Aut(T^rX) = Isom(X)$ for any compact real-analytic Riemannian manifold X of any $r \leq r_{max}(X)$. For the non-compact centers, the only result obtained so far is contained in [8] [9] and [10]. The authors prove the rigidity statement for Grauert tubes T^rX over any symmetric space X of $r < r_{max}(X)$ provided that T^rX is not covered by the ball.

In the compact case, one of the key observation toward the rigidity of Grauert tubes is the following: an automorphism f of the Grauert tube T^rX comes from the differential of an isometry of the Riemannian manifold (X, g) if and only if f keeps the center X invariant, i.e., $f(X) = X$. We will show in the following theorem that Grauert tubes over non-compact centers share the same property.

Theorem 4.1. *Let $f \in Aut(T^rX)$, $r \leq r_{max}$. Then $f = du$ for some $u \in Isom(X)$ if and only if $f(X) = X$.*

Proof. One direction is clear. Suppose $f(X) = X$. We denote the Monge-Ampère

model over (X, g) as (Ω, X, g, ρ) which means $\Omega = T^r X$ is the Grauert tube over X of radius r; ρ is a non-negative strictly plurisubharmonic function satisfying the homogeneous complex Monge-Ampère equation on $\Omega \backslash X$; the zero set of ρ is exactly X and $(g_{ij}) = 2(\rho_{i\bar{j}}|_X)$ is the Riemannian metric on X induced from the Kähler form $i\partial\bar{\partial}\rho$. The automorphism f maps the Monge-Ampère model (Ω, X, g, ρ) onto the Monge-Ampère model $(\Omega, X, k, \rho \cdot f^{-1})$ where k is the Riemannian metric on X induced from the Kähler form $i\partial\bar{\partial}(\rho \cdot f^{-1})$. That is to say that Ω is a Grauert tube of radius r over the Riemannian manifold (X, k) as well. Let

$$u = f|_X : (X, g) \to (X, k)$$

be the restriction of f to X. The map u is an isometry from (X, g) to (X, k). Actually, it is an isometry of (X, g) since the Grauert tube $\Omega = T^r(X, g) = T^r(X, k)$ is the collection of tangent vectors of X of length less than r under both metrics g and k. This forces the metric k to be equal to the metric g.

Hence $u \in Isom(X, g)$ and $du \in Aut(\Omega)$. The automorphism $du \cdot f^{-1}$ of Ω is the identity on a maximal totally real submanifold X. Therefore

$$f = du$$

on the whole Ω. $\square$

It is in general easier to deal with simply-connected manifolds. We will assume all of the manifolds X are simply-connected for proofs in Section 7. Before making this assumption, we claim that it won't do any hurt for the general situations. Notice that for any given X, we could always lift it to its universal covering $\tilde{X}$ which is simply-connected. Let's denote the Grauert tube $T^r X$ as Ω. Then the Grauert tube $T^r \tilde{X} = \tilde{\Omega}$ is the universal covering of Ω. We have the following relation on the rigidity of $\tilde{\Omega}$ and Ω.

Lemma 4.2. *Let $X, \tilde{X}, \Omega$ and $\tilde{\Omega}$ be as above. Suppose there is a unique Grauert tube representation for $\tilde{\Omega}$, then there is a unique Grauert tube representation for Ω*

Proof. If $\Omega = T^r X = T^r Y$. Then $\tilde{\Omega} = T^r \tilde{X} = T^r \tilde{Y}$ has two Grauert tube representations. Thus $\tilde{X} = \tilde{Y}$. On the other hand, $X = \tilde{X}/\Gamma$ for some $\Gamma \subset Isom(\tilde{X})$. Hence $\Omega = \tilde{\Omega}/\Gamma$. It follows that $Y = \tilde{Y}/\Gamma = \tilde{X}/\Gamma = X$. $\square$

Notice that the universal covering of a homogeneous Riemannian manifold is homogeneous (c.f. Theorem 2.4.17 [17]). Hence the rigidity of Grauert tubes over

homogeneous Riemannian spaces automatically holds once we prove it for simply-connected homogeneous Riemannian spaces.

5. The complete hyperbolicity of T^rX.

It was proved by Sibony on Theorem 3 of [16] that a complex manifold admitting a bounded strictly plurisubharmonic function is hyperbolic. This implies that every Grauert tube of finite radius is a hyperbolic manifold. When the center is compact and the radius $r < r_{max}$, the Grauert tube T^rX is a strictly pseudoconvex domain sitting inside the Stein manifold $T^{r_{max}}X$ and hence is complete hyperbolic. There is no such generality for non-compact cases. So far, the only known characterization is over co-compact manifolds.

When X is co-compact, i.e., X/Γ is compact for some $\Gamma \subset Isom(X)$, the Grauert tube T^rX is simply the universal covering of $T^r(X/\Gamma)$ and hence is completely hyperbolic in case $r < r_{max}$. We will show in this section that a Grauert tube constructed over a homogeneous Riemannian manifold of radius less than the maximal is complete hyperbolic.

A sequence $f_j \in Hol(N, W)$ is said to be *compactly divergent* if for each compact subset $K \subset N$ and each compact $K' \subset W$ there is a $j_0 = j_0(K, K')$ such that the set $f_j(K) \cap K'$ is empty for each $j \geq j_0$. A complex manifold W is said to be *taut* if each sequence in $Hol(N, W)$ contains a subsequence that either converges to an element of $Hol(N, W)$ in the compact open topology, or diverges compactly. It is known that a complete hyperbolic manifold is taut and a taut manifold is hyperbolic.

A Riemannian manifold is said to be *homogeneous* if its isometry group acts transitively on it. It is a classical result that a Riemannian manifold (X, g) is complete if and only if every bounded subset is relatively compact in X. Let d denote the distance for the metric g in X and $B(x, r) = \{y \in X : d(x, y) < r\}$ denote the ball of radius r centered at $x \in X$.

Lemma 5.1. *Let X be a non-compact homogeneous Riemannian manifold, $x \in X$. Then for any $R > 0$ there exist $y \in X$ such that $d(x, y) > R$.*

Proof. Suppose there exist a $x \in X$ and a $R > 0$ such that $d(x, y) < R$ for all $y \in X$. Then $X \subset B(x, R)$. Since $B(x, R)$ is a bounded subset of X; it is relatively

compact in X. Thus, $\overline{B(x,R)}$ is a compact subset of X and

$$\overline{B(x,R)} \subset X \subset B(x,R)$$

which implies that $X = \overline{B(x,R)} = B(x,R)$ is compact, a contradiction. $\square$

Let $\Omega = T^r X$ be a Grauert tube of $r < r_{max}$ and let d_K denote the Kobayashi distance on Ω. The boundary of Ω contains two parts. The first part is consisted of smooth strictly pseudoconvex points $\{(x,v) : \ v \in T_x X, \ |v| = r\}$. The second part comes from the boundary of the Riemannian manifold X; it is the set $\{(x,v) : v \in T_x X, \ |v| < r, x \in B^C(p,R), p \in X, \forall R > 0\}$.

From Lemma 2.2 we know that for any $p \in \Omega$, $d_K(p,q) \to \infty$ as q approaches a smooth strictly pseudoconvex boundary point. A metric is complete means that every bounded subset is relatively compact. To show the complete hyperbolicity of Ω, it is therefore sufficient to show that $d(p,y) \to \infty$ as y goes to the boundary of X.

Theorem 5.2. *Let X be a homogeneous Riemannian manifold, $\Omega = T^r X$ be a Grauert tube of $r < r_{max}$. Then Ω is complete hyperbolic.*

Proof. The statement is clear for compact X by the existence of a strictly plurisubharmonic exhaustion function. We shall prove the non-compact case.

For $p \in X$, let $S(p,n) = \{x \in X : d(p,x) = n\}$ be the n-sphere in X around p. By Lemma 5.1, $S(p,n) \neq \emptyset$ for all $n \in \mathbb{N}$. Picking a point $p_2 \in S(p,2)$, there exists a unique minimal geodesic $\gamma_2(t)$, parametrized by the arclength, joining p to p_2. The geodesic $\gamma_2(t)$ also serves as the minimal geodesic for any two points $\gamma_2(a)$ and $\gamma_2(b)$, $0 \leq a \leq b \leq 2$. Then $p_1 := \gamma_2(1) \in S(p,1)$ and $d(p_1,p_2) = 1$. Let

$$\mu_1 = \min\{d_K(p,x) : x \in S(p,1)\} > 0.$$

Since $p_1 \in S(p,1)$, it follows that

$$d_K(p,p_1) \geq \mu_1. \tag{5.1}$$

Since X is homogeneous, there exist $g_j \in Isom(X)$ such that

$$g_j(p_j) = p, \quad j = 1,2.$$

Hence

$$d(p, g_1(p_2)) = d(g_1(p_1), g_1(p_2)) = d(p_1,p_2) = 1.$$

This implies that $g_1(p_2) \in S(p,1)$ and therefore

$$d_{\scriptscriptstyle K}(p, g_1(p_2)) \geq \mu_1. \tag{5.2}$$

As the Kobayashi distance is biholomorphically invariant and $g_j \in Isom(X) \subset Aut(\Omega)$, we have

$$d_{\scriptscriptstyle K}(p_1,p_2) = d_{\scriptscriptstyle K}(g_1(p_1), g_1(p_2)) = d_{\scriptscriptstyle K}(p, g_1(p_2)) \geq \mu_1. \tag{5.3}$$

Let $B_{\scriptscriptstyle K}(x,\mu_1)$ denote the restriction of the Kobayashi ball of radius μ_1 to X, i.e.,

$$B_{\scriptscriptstyle K}(x,\mu_1) = \{y \in X : d_{\scriptscriptstyle K}(x,y) < \mu_1\}.$$

Then $B_{\scriptscriptstyle K}(p,\mu_1) \subset B(p,1)$. Therefore we have, for $j = 1,2$,

$$\begin{aligned} B_{\scriptscriptstyle K}(p_j,\mu_1) &= B_{\scriptscriptstyle K}(g_j^{-1}(p),\mu_1) = g_j^{-1} B_{\scriptscriptstyle K}(p,\mu_1) \\ &\subset g_j^{-1} B(p,1) = B(g_j^{-1}(p),1) = B(p_j,1). \end{aligned} \tag{5.4}$$

For $x \in B(p,1)$, $d(p_2,x) \geq d(p_2,p) - d(p,x) \geq 1$. Hence

$$B(p,1) \cap B(p_2,1) = \emptyset. \tag{5.5}$$

(5.4) and (5.5) immediately imply that

$$B_{\scriptscriptstyle K}(p,\mu_1) \cap B_{\scriptscriptstyle K}(p_2,\mu_1) = \emptyset. \tag{5.6}$$

Thus, $d_{\scriptscriptstyle K}(p,p_2) \geq 2\mu_1$. Since p_2 could be taken from arbitrary points in $S(p,2)$, we have

$$d_{\scriptscriptstyle K}(p,q) \geq 2\mu_1 \quad \text{for all } q \in S(p,2). \tag{5.7}$$

Repeating the same precess by taking $q_4 \in S(p,4)$, the corresponding minimal geodesic joining p and q_4 be γ_4, $q_2 = \gamma_4(2) \in S(p,2)$ and

$$\mu_2 = \min\{d_{\scriptscriptstyle K}(p,x) : x \in S(p,2)\}.$$

By (5.7), $\mu_2 \geq 2\mu_1$. We get

$$d_{\scriptscriptstyle K}(p,q) \geq 2\mu_2 \geq 2^2\mu_1, \quad \text{for all } q \in S(p,2^2). \tag{5.8}$$

By inductive argument, we conclude that

$$d_{\scriptscriptstyle K}(p,q) \geq 2^n\mu_1, \quad \text{for all } q \in S(p,2^n). \tag{5.9}$$

Finally, we claim that Ω is complete hyperbolic, i.e., every bounded subset is relatively compact in Ω or equivalently,

$$\lim_{q \to \partial\Omega} d_K(z,q) = \infty. \tag{5.10}$$

It was already shown on Lemma 2.2 that if q goes to a smooth strictly pseudoconvex boundary point then (5.10) holds, so it is sufficient to check those $q \in X$ staying away from any bounded set in X. If q stays outside of $B(p, 2^n)$ we conclude that $d_K(p,q) \to \infty$ as $n \to \infty$. Moreover, for any $z \in \Omega$,

$$d_K(q,z) \geq d_K(q,p) - d_K(p,z) \to \infty$$

as well. Hence the Kobayashi metric d_K is complete. $\square$

6. The equality of $Aut_0(T^rX) = Isom_0(X)$.

Since a Grauert tube of finite radius is hyperbolic, its automorphism group is a Lie group. It is clear from the construction of Grauert tubes that the differential du is an automorphism of T^rX for any $u \in Isom(X)$. Hence the isometry group $Isom(X)$ is a Lie subgroup of $Aut(T^rX)$. Without any ambiguity, we also use the same symbol u to represent its differential du in $Aut(T^rX)$.

From now on we shall assume that X is a real-analytic Riemannian manifold such that the adapted complex structure exists on T^rX. Let

$$T_p^rX = \{(p,v) : v \in T_pX, \ |v| < r\}$$

be the fiber through the point $p \in X$ and σ be the anti-holomorphic involution

$$\sigma : T^rX \to T^rX, \ \ (p,v) \to (p,-v).$$

When $r < r_{max}$, the boundary ∂T_p^rX is consisted of smooth strictly pseudoconvex points since it is locally defined by $\{\rho^{-1}(r^2)\}$.

Let I denote the isometry group of X and G denote the automorphism group of T^rX. Then the group

$$\hat{G} := G \cup \sigma \cdot G$$

is again a Lie group acting on T^rX. Let $\hat{\mathcal{G}}$ be the Lie algebra of $\hat{G}$. Each $\xi \in \hat{\mathcal{G}}$ could be viewed as a vector field in T^rX; $\xi(z) = \frac{d}{dt}|_{t=0}(exp\ t\xi)(z)$.

Lemma 6.1. *For each $\xi \in \hat{\mathcal{G}}$, there is an $\eta \in \hat{\mathcal{G}}$ corresponding to ξ such that for any $t \in \mathbb{R}$, $\exp t\eta : T_p^r X \to T_p^r X, \ \forall p \in X$.*

Proof. Define $\eta \in \hat{\mathcal{G}}$ as

$$\eta = \xi - \frac{d}{dt}|_{t=0}(\sigma \cdot (exp\ t\xi)).$$

In local coordinates, $\sigma(z) = \bar{z}$. Then

$$\eta(iy) = 2i \text{ Im } \xi(iy).$$

It follows that η is tangent to the fiber $T_p^r X$ and the result follows. $\square$

Fix a point $p \in X$ and take U to be a small neighborhood of p in X. Take

$$D = \{(x, v) : x \in U, v \in T_x U, |v| < r\}$$

be a domain in $T^r X$ with the induced complex structure from $T^r X$. The domain D could be equipped with the metric d induced from the Kobayashi metric d_K of $T^r X$, i.e., define the metric $d(z, w) := d_K(z, w), \forall z, w \in D$.

For a fixed $\eta \in \hat{\mathcal{G}}$ as in the statement of the above lemma, we consider the restriction maps $exp\ (t\eta)|_D$ and set

$$\mathcal{F} = \{exp\ (t\eta)|_D; t \in \mathbb{R}\} \subset C(D, T^r X),$$

where $C(D, T^r X)$ denotes the set of all continuous maps from D to $T^r X$.

Lemma 6.2. *Suppose $T^r X$ is not biholomorphic to the ball and $r < r_{max}$, then $\mathcal{F}$ is a compact subset of $C(D, T^r X)$.*

Proof. It is clear that $\mathcal{F}$ is closed in $C(D, T^r X)$. By Lemma 6.1, $f(D) \subset D$ for all $f \in \mathcal{F}$. As d_K is an invariant metric and $exp\ (t\eta) \in Aut(T^r X)$, we have

$$d(exp\ (t\eta)(z), exp\ (t\eta)(w)) = d_K(exp\ (t\eta)(z), exp\ (t\eta)(w)) = d_K(z, w) = d(z, w)$$

for all $z, w \in D$ and $t \in \mathbb{R}$. This shows that $\mathcal{F}$ is equicontinuous. We then claim that for every $z \in D$, the set $\mathcal{F}(z) := \{exp\ (t\eta)(z) : t \in \mathbb{R}\}$ is relatively compact in $T^r X$. Suppose not, for some $z = (p, v) \in D$ the set of points $\{exp\ (t\eta)(z)\}$ approach to the boundary of $T_p^r X$ which is consisted of smooth strictly pseudoconvex points. By Theorem 3.1, this forces $T^r X$ to be the ball which is a contradiction. Therefore $\mathcal{F}(z)$ is a relative compact subset of $T^r X$. By the Ascoli theorem (c. f. [19]), $\mathcal{F}$ is compact in $C(D, T^r X)$. $\square$

Lemma 6.3. *For each $\xi \in \hat{\mathcal{G}}$, the vector field ξ is tangent to X.*

Proof. By lemmas 6.1 and 6.2, $\mathcal{F}$ is a connected compact Lie group acting symmetrically on each fibre $T^r_p X$. The action is symmetric as defined on §4 of [10] since if $g \in \mathcal{F}$ then $\sigma \cdot g \cdot \sigma(y) = -g(-y)$. Applying Theorem 4.1 of [10], every $p \in X$ is a fixed point of any $g \in \mathcal{F}$. From the construction of η in Lemma 6.1, this implies that

$$\text{Im } \xi(p) = -\frac{i}{2}\eta(p) = 0, \ \ \forall p \in X.$$

Therefore ξ is tangent to X. $\square$

Denote the identity component of $Isom(X)$ by I_0 and the identity component of $Aut(T^r X)$ by G_0. We prove the following for Grauert tubes over any real-analytic Riemannian manifold.

Theorem 6.4. *Let X be a real-analytic Riemannian manifold such that $r_{max}(X) > 0$. Then $Aut_0(T^r X) = Isom_0(X)$ for any $r < r_{max}$ provided that $T^r X$ is not covered by the ball.*

Proof. There exist a neighborhood V of 0 in $\hat{\mathcal{G}}$ and a neighborhood U of id in $\hat{G}_0$ such that the exponential map is a diffeomorphism from V to U. That is, for every $f \in U$, there exists an $\xi \in V$ such that $f = exp\ \xi$. By Lemma 6.3, $f(X) \subset X$. As X is a closed submanifold and f is a diffeomorphism of $T^r X$, $f(X)$ is then a closed submanifold of the connected manifold X of the same dimension, which implies that $f(X) = X$.

By Theorem 4.1, $f \in I_0$. Hence $U \subset I_0 \subset \hat{G}_0$ and we conclude that the manifolds I_0 and $\hat{G}_0$ have the same dimension. Since I_0 is a Lie subgroup of $\hat{G}_0$, I_0 is a closed submanifold of $\hat{G}_0$, therefore $I_0 = \hat{G}_0$. On the other hand, it is clear from the definition of $\hat{G}$ that $\hat{G}_0 = G_0$. We conclude that $I_0 = G_0$. $\square$

7. The rigidity of Grauert tubes over homogeneous centers.

For compact X, Burns-Hind [2] have proved that the isometry group $Isom(X)$ of X is isomorphic to the automorphism group $Aut(T^r X)$ of the Grauert tube for any radius $r \leq r_{max}$. Burns [1] also proved that this isomorphism is equivalent to the uniqueness of the center.

For the non-compact cases, the only known rigid Grauert tubes are Grauert tubes constructed over locally symmetric spaces of $r < r_{max}$ in [10] provided that

the Grauert tubes are not covered by the ball. In [9], the author further proved the uniqueness of the center also holds for the above non-compact cases.

We will prove in this section that not only the rigidity but also the uniqueness holds for Grauert tubes over non-compact homogeneous Riemannian manifolds of $r < r_{max}$; the only exception occurs when the Grauert tube is biholomorphic to the ball. It was proved by the author in [8] that T^rX is biholomorphic to $B^n \subset \mathbb{C}^n$ if and only if X is the real hyperbolic space $\mathbb{H}^n$ of curvature -1 and the radius $r = \frac{\pi}{4}$. (The Monge-Ampère solution ρ in this article is half of the ρ in [8], hence the radius changes.) Apparently, the automorphism group of B^n is much larger than the isometry group of $\mathbb{H}^n$.

We assume in this section that X is a simply-connected Riemannian homogeneous space and $\Omega = T^rX$ is the Grauert tube over X of radius r. Recall that a Riemannian manifold X is homogeneous if the isometry group acts transitively on X. Denote $I = Isom(X)$ and $G = Aut(T^rX)$; I_0 and G_0 the corresponding identity components. Notice that the homogeneity immediately implies that the $X = I_0(p)$ for any $p \in X$.

For any given $f \in G$, the set $Y = f(X)$ equipped with the push-forward metric coming from X is a homogeneous center of the Grauert tube Ω. We claim that Y crosses through every fiber T^r_pX.

Lemma 7.1. *$Y \cap T^r_pX \neq \emptyset$ for any $p \in X$.*

Proof. Since X is a connected homogeneous space, the fact that $I_0 = G_0$ is a normal subgroup of G would imply that for any $p \in X$, we have

$$Y = f(X) = f(I_0 \cdot p) = I_0 \cdot f(p) \tag{7.1}$$

is an n-dimensional I_0-orbit as well. Let

$$\pi : \Omega \to X, \ \ \pi(p, v) = p$$

be the projection map. It is clear that π is I_0-equivariant since for any $(x, v) \in \Omega$ and $g \in I_0$, we have

$$\pi(g \cdot (x, v)) = \pi(g \cdot x, g_* v) = g \cdot x = g \cdot \pi(x, v).$$

By (7.1),

$$\pi(Y) = \pi(I_0 \cdot f(p)) = I_0 \cdot \pi(f(p)) = X.$$

Therefore Y intersects every fiber T^r_pX. The lemma is proved. □

The Lie group I is a subgroup of G. We consider the quotient space G/I and examine the index of this coset space.

Proposition 7.2. *The index of I in G is finite.*

Proof. Let $\{g_j I\}, g_j \in G$, be a sequence in the coset space G/I. By Lemma 7.1, $g_j(X)$ has non-empty intersection with the fiber $T^r_p X$. Take a point

$$q_j \in g_j(X) \cap T^r_p X.$$

Since $g_j^{-1}(q_j) \in X$ and I acts transitively on X, there exists a $h_j \in I$ sending the point $p \in X$ to $g_j^{-1}(q_j) \in X$. Let

$$f_j = g_j \cdot h_j \in G$$

then $f_j(p) = q_j \in T^r_p X$ for all j. Since the boundary of $T^r_p X$ is consisted of smooth strictly pseudoconvex points and the Grauert tube Ω is not the ball, the generalized Wong-Rosay theorem in §3 implies that no subsequence of $\{f_j\}$ could diverge compactly. By Theorem 5.2, Ω is a taut manifold which asserts that there exists a subsequence of $\{f_j\}$ converges to some $f \in G$ in the topology of G. Hence, some subsequence of $\{g_j I\}$ converges to gI in the topology of G/I. Thus, G/I is compact. Since I and G have the same identity components, I is open in G. The compactness implies that the index of G/I is finite. $\square$

Suppose the Grauert tube $\Omega = T^r X$ could be represented as a Grauert tube of the same radius of another center Y, i.e., $\Omega = T^r Y$. We use the notations (X, σ) and (Y, τ) to indicate that the two anti-holomorphic involutions of the Grauert tube Ω with respect to the centers X and Y are σ and τ respectively. We call (Y, τ) a *homogeneous center* if Y is a homogeneous space as well. Regrouping the terms, we have the following relation:

$$\begin{aligned} \tau \cdot (\sigma \cdot \tau)^{2l} &= (\sigma \cdot \tau)^{-l} \cdot \tau \cdot (\sigma \cdot \tau)^{l} \\ \tau \cdot (\sigma \cdot \tau)^{2l-1} &= (\tau \cdot (\sigma \cdot \tau)^{l-1}) \cdot \sigma \cdot (\tau \cdot (\sigma \cdot \tau)^{l-1})^{-1} \end{aligned} \tag{7.2}$$

Proposition 7.3. *Let (Y, τ) be a homogeneous center of Ω and let k be the least positive integer such that $(\sigma \cdot \tau)^k \in I$. Then k is odd and $(\sigma \cdot \tau)^k = id$.*

Proof. Since the index of I in G is finite there exists the least integer k with $(\sigma \cdot \tau)^k \in I$. Write $(\sigma \cdot \tau)^k = du$ for some $u \in I$. By (7.2),

$$\sigma \cdot du = \tau \cdot (\sigma \cdot \tau)^{k-1}$$

is an anti-holomorphic involution in Ω with fixed point set Z where $Z = \tau \cdot (\sigma \cdot \tau)^{\frac{k-2}{2}}(X)$ when k is even; $Z = (\sigma \cdot \tau)^{-\frac{k-1}{2}}(Y)$ when k is odd.

$(Z,\ \sigma\cdot du)$ is a homogeneous center, with the pushed forward metric coming from X or Y, of the Grauert tube Ω. Suppose there is $z \in Z - X,\ z = (x, v) \in T_x X, v \neq 0$. Since z is a fixed point of $\sigma \cdot du$,

$$(x, v) = \sigma \cdot du(x, v) = \sigma \cdot (u(x), u_* v) = (u(x), -u_* v).$$

We have $u(x) = x,\ v = -u_* v$. Hence the whole interval

$$L = \{(x, \alpha v) \in T^r_x X : |\alpha v| < r\}$$

is fixed by $\sigma \cdot du$ which implies that $Z \supset L$. Since Z is a homogeneous space, there exist $\{f_j\} \in Isom(Z) \subset G$ such that

$$f_j(x) = (x, \frac{jr}{(j+1)|v|} v),\ j \in \mathbb{N}.$$

Then $f_j(x) \to (x, \frac{r}{|v|} v)$, a smooth strictly pseudoconvex boundary point of Ω, as $j \to \infty$. By Theorem 3.1, Ω is then biholomorphic to the ball, a contradiction. Therefore $Z = X,\ \sigma \cdot du = \sigma$ and hence $du = (\sigma \cdot \tau)^k = id$.

Suppose $k = 2l$ is even. By (7.2), $\sigma = (\sigma \cdot \tau)^l \cdot \sigma \cdot (\sigma \cdot \tau)^{-l}$. Thus,

$$\sigma \cdot (\sigma \cdot \tau)^l = (\sigma \cdot \tau)^l \cdot \sigma.$$

The n–dimensional closed submanifold $(\sigma \cdot \tau)^l(X)$ is the fixed point set of the anti-holomorphic involution σ. By the uniqueness of the fixed point set, $(\sigma \cdot \tau)^l(X) = X$. By Theorem 4.1, $(\sigma \cdot \tau)^l \in I$ which is a contradiction since $l < k$. □

We also prove a proposition similar to the Theorem 2bis of [1].

Proposition 7.4. *For any homogeneous center (Y, τ) of Ω, there exists an $f \in G$ such that $f(X) = Y$. Furthermore, there are at most a finite number of homogeneous centers in Ω.*

Proof. Let $k = 2l + 1$ be the odd integer as stated in Proposition 7.3 and let $f := (\sigma \cdot \tau)^l \in G$. The condition $(\sigma \cdot \tau)^k = id$ holds if and only if

$$\tau = (\sigma \cdot \tau)^{2l+1} \cdot \tau = (\sigma \cdot \tau)^l \cdot \sigma \cdot (\sigma \cdot \tau)^{-l} = f \cdot \sigma \cdot f^{-1}.$$

Thus, $f(X)$ is the fixed point set of τ, i.e., $f(X) = Y$.

Since G/I has finite index h, there exist $\{g_j \in G\}_{j=1}^h$ such that G is the disjoint union of $g_j I, j = 1, \cdots, h$. Then for any $\eta \in G, \eta(X) \in \{g_j(X)\}_{j=1}^h$ where $(g_j(X),\ g_j \cdot \sigma \cdot g_j^{-1})$ are homogeneous centers of Ω. Suppose there exists a homogeneous center $(W, \hat{\tau})$ other than $\{(g_j(X), (g_j \cdot \sigma \cdot g_j^{-1})\}_{j=1}^h$, then there is an $\hat{f} \in G$ such that $\hat{f}(X) = W$, a contradiction. □

Lemma 7.5. *$Isom(X) = Aut(\Omega)$ if and only if there is a unique homogeneous center (X, σ) for Ω.*

Proof. Suppose there exists another homogeneous center (Y, τ), then we could find $f \in G$ such that $Y = f(X)$ by Proposition 7.4. The condition $I = G$ implies that $Y = X$.

Suppose there exists $\zeta \in G \backslash I$ sending (X, σ) to another homogeneous center $(\zeta(X),\ \zeta \cdot \sigma \cdot \zeta^{-1})$. By the assumption, $\zeta(X) = X$. Thus, by Theorem 4.1, $\zeta \in I$, a contradiction. $\square$

Recall that a Grauert tube $\Omega = T^r X$ is said to have a unique center X if this is the unique Grauert tube representation for Ω, i.e., there is no other Y such that $\Omega = T^r Y$. Finally we prove the main theorem of this paper, the rigidity and the uniqueness of the center.

Theorem 7.6. *Let X be (or be a quotient manifold of) a real-analytic homogeneous Riemannian manifold. Let $\Omega = T^r X$ be a Grauert tube of radius $r < r_{max}$ such that Ω is not covered by the ball. Then $Isom(X) = Aut(\Omega)$ and Ω has a unique center.*

Proof. Since the proof of Proposition 7.2 works through when replacing I by G_0, it is easy to see that the index of G/G_0 is finite.

Following the same arguments in the proof of Lemma 7.4 in [10]. We are able to construct an $\hat{G}$-invariant strictly plurisubharmonic non-negative function

$$\psi(z) = \sum_{j=1}^{k} \rho(g_j(z))$$

in Ω where $\{g_1, \ldots, g_k\} \in G$ so that $G/G_0 = \{g_j G_0 : j = 1, \ldots, k\}$. As G_0 acts transitively on X, the tangent space $T_z(T^r X)$ could be decomposed as, for any $z \in T^r X$,

$$T_z(T^r X) = T_z(G_0 \cdot z) + T_z(T^r_{\pi(z)} X).$$

Since ψ is constant in $G_0 \cdot z$, every critical point of the function $f_z := \psi|_{T^r_{\pi(z)} X}$ is a critical point of ψ and every critical point of ψ occurs at the critical points of the functions f_z.

As ψ is strictly plurisubharmonic, the above decomposition implies that the real Hessian of f_z is positive definite on the tangent space $T_z(T^r_{\pi(z)} X)$. Since f_z is proper on the fiber, it follows that there is exactly one critical point of f_z which turns out

to the the minimal point. Since $\psi \cdot \sigma = \psi$, the minimum of f_z occurs at $\pi(z)$. That is to say that the set of critical points of ψ is exactly X.

Suppose there is another homogeneous center Y. By proposition 7.4, $Y = F(X)$ for some $F \in G$. Since ψ is G-invariant, every point of Y is a critical point of ψ. Therefore, $Y \subset X$. We conclude that $X = Y$ and $Isom(X) = Aut(\Omega)$ is proved following from Lemma 7.5.

We then prove that there is exactly one center in Ω. Suppose there is a non-homogeneous center W of the Grauert tube Ω. This W has to be connected and $dim\ W = dim\ X$.

Let $z = (x, v) \in W$. Since $Isom_0(W) = Aut_0(\Omega) = Isom_0(X)$, for any $g \in Isom_0(W)$ there exists $u \in Isom_0(X)$ such that

$$g \cdot (x, v) = dg \cdot (x, v) = du \cdot (x, v) = (u(x), u_* v).$$

Let g run through the Lie group $Isom_0(W)$, the corresponding u will then go through $Isom_0(X)$, we obtain

$$W \supset Isom_0(W) \cdot (x, v) = Isom_0(X) \cdot (x, v).$$

By the homogeneity of X, the left hand side is a closed submanifold of dimension $\geq n$. Therefore, $Isom_0(W) \cdot (x, v)$ is a closed submanifold of dimension n in W. As W is connected, we conclude that $W = Isom_0(W) \cdot (x, v)$ and hence W is homogeneous, a contradiction.

□

Remark.

(1) Since every symmetric space is homogeneous, results in this article cover those in [9] and [10].

(2) It was shown in [3] that Grauert tubes over non-compact symmetric spaces of rank one of the maximal radius are not rigid.

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